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On Operators Commuting with Toeplitz Operators Modulo the Compact Operators

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We prove that an operator on H^2 of the disc commutes modulo the compacts with all analytic Toeplitz operators if and only if it is a compact perturbation of a Toeplitz operator with symbol in $H^\infty + C$. Consequently, the essential commutant of the whole Toeplitz algebra is the algebra of Toeplitz operators with symbol in QC. The image in the Calkin algebra of the Toeplitz operators with symbol in $H^\infty + C$ is a maximal abelian algebra. These results lead to a characterization of automorphisms of the algebra of compact perturbations of the analytic Toeplitz operators.

Johnson and Parrot [7] showed that if M is an abelian von Neumann algebra on a Hilbert space H, M' is its commutant, and $\mathscr{LC}(H)$ is the ideal of compact operators on H, then the essential commutant of M is $M' + \mathscr{LC}(H)$. About the same time, Sarason [8] showed that a Toeplitz operator T_g on H^2 of the unit circle commutes modulo the compacts with all analytic Toeplitz operators if and only if g is in $H^\infty + C$. Here C denotes the space of continuous functions on the unit circle. From this, Douglas [6] showed that the essential center of the Toeplitz algebra is the algebra of Toeplitz operators with symbol in $QC = H^\infty + C \cap \overline{H^\infty + C}$. Douglas [4] raised the natural question of which operators in $\mathscr{L}(H^2)$ essentially commute with all Toeplitz operators.

In this paper we prove the following theorem, which gives a complete answer to this question.

THEOREM 1. *An operator S on H^2 commutes modulo compacts with all analytic Toeplitz operators if and only if $S = T_g + K$, where g is in $H^\infty + C$ and K is compact.*

From this we get two immediate corollaries.

COROLLARY 1. *The essential commutant of the Toeplitz algebra $\mathscr{T}(L^\infty)$ is $\mathscr{T}(QC)$.*

COROLLARY 2. *The image of $\mathscr{T}(H^\infty + C)$ in the Calkin algebra is a maximal abelian algebra.*

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As far as the author knows, Corollary 2 gives the first concrete example of a maximal abelian algebra in the Calkin algebra which is not an image of a maximal abelian von Neumann algebra. (That the latter is an example is a consequence of [7].)

The author would like to point out that, although the techniques of this paper are quite different from those used in [7], several important conceptual ideas for the proof were gleaned from the Johnson and Parrot paper.

Let $\mathscr{D}$ be the unit disc, and let $\partial\mathscr{D}$ be its boundary. Let H^2 be the subspace of $L^2 = L^2(\partial\mathscr{D})$ of those functions with all negative Fourier coefficients equal to zero. Let M_f denote the operator on L^2 of multiplication by the L^∞ function f. The Toeplitz operator T_f with symbol f is the compression of M_f to H^2. If A is a subset of L^∞, let $\mathscr{T}(A)$ be the norm closed algebra generated by $\{T_f : f \in A\}$. Let $\mathscr{L}(H^2)$ be the bounded operators on H^2, and let $\mathscr{LC}(H^2)$ be its ideal of compact operators. Let $\pi: \mathscr{L}(H^2) \to \mathscr{L}(H^2)/\mathscr{LC}(H^2)$ be the canonical homomorphism onto the Calkin algebra. If S is an operator, let $D(X) = XS - SX$ be the derivation on $\mathscr{L}(H^2)$ induced by S.

We will prove the following theorem, which is somewhat stronger than Theorem 1.

THEOREM 2. *If an operator S in $\mathscr{L}(H^2)$ is not the sum of a Toeplitz operator and a compact operator, then there is a function $h \in H^\infty$ such that $T_hS - ST_h$ is not compact. The function h may be taken to have at most one discontinuity.*

The proof will follow from a series of lemmas, but first we will outline the main ideas. If $T_zS - ST_z$ is not compact, we can take $h = z$. So for the remainder of the proof, we will assume that S essentially commutes with T_z. From this, it follows that S commutes modulo compacts with every Toeplitz operator with continuous symbol. We show that there is a subsequence Λ of the positive integers and a function f in L^∞, such that in the w^* topology on $\mathscr{L}(H^2)$,

$$T_f = \lim_{n\in\Lambda} T_{\bar{z}^n} S T_{z^n}.$$

Next, we find a countable collection of disjoint closed intervals $\{\chi_n\}$ of the unit circle such that $\| T_{\chi_n}S - T_{\chi_n f} \| > \delta > 0$. (Here we take the liberty of denoting both χ_n and its characteristic function by the same symbol.) Combining the two preceding results, we obtain functions p_n in H^∞ such that $\| D(T_{p_n})\| > \delta$ and so that the partial sums of the series $\sum p_n$ are uniformly bounded. It follows that T_{p_n} and $D(T_{p_n})$ converge strongly to zero, so we are able to extract a subsequence Γ such that $h = \sum_{n\in\Gamma} p_n$ is in H^∞ with the operators $D(T_{p_n})$ almost mutually orthogonal. This will allow us to conclude that $D(T_h)$ is not compact. By choosing the functions p_n so that their closed supports cluster at only one point, we ensure that h has only one discontinuity.

Let Σ be the net of inner functions ordered by divisibility. If ω is an inner function, let $\sigma_\omega = S - T_{\bar{\omega}} S T_\omega$. Let $\sigma_n = \sigma_{z^n}$. Consider the sequence $\{\sigma_n\}$ and the net $\{\sigma_\omega\}$. Both are norm bounded, and hence lie in a w^* compact set. Consequently, they have w^* limit points.

LEMMA 1. *Let S be in $\mathscr{L}(H^2)$ with $T_z S - S T_z$ compact, and let S' be a w^* limit point of the sequence $\{\sigma_n\}$. Then,*

(1) $T_g S - S T_g$ *is compact for all continuous functions* g.

(2) $S' = w^* \lim_{n \in \Lambda} \sigma_n$ *for a subsequence* Λ *of* $\mathbb{N}$.

(3) $S - S' = T_f$ *for some* f *in* L^∞.

(4) $S' - T_{\bar{b}} S' T_b = \sigma_b$ *for all continuous inner functions* b.

(5) $w^* \lim_{n \in \Lambda} T_{\bar{z}^n} T_g S T_{z^n} = T_{gf}$ *for all continuous functions* g.

The operator S' is compact if and only if S is a compact perturbation of a Toeplitz operator. In this case, in the norm topology,

$$\lim_{n \to \infty} \sigma_n = S' \qquad \text{and} \qquad \lim_{n \to \infty} T_{\bar{z}^n} S' T_{z^n} = 0.$$

Let $S - T_{\bar{\omega}} S T_\omega$ be compact for all inner functions $\omega \in \Sigma$, and let $S' = w^* \lim_{\omega \in \Lambda} \sigma_\omega$ for a subnet Λ of Σ. Then,

(4′) $S' - T_{\bar{\omega}} S' T_\omega = \sigma_\omega$ *for all inner functions* ω.

(5′) $w^* \lim_{n \in \Lambda} T_{\bar{\omega}} T_h S T_\omega = T_{hf}$ *for all* h *in* L^∞.

If $S = T_f + X$ is in the Toeplitz algebra, where X belongs to the commutator ideal of $\mathscr{T}(L^\infty)$, then

$$\lim_{\omega \in \Sigma} \sigma_\omega = X \qquad \textit{in norm}.$$

Proof. The set of operators $\{\sigma_n\}$ lies in the ball of radius $2\|S\|$, which is w^* compact and metrizable. Hence the w^* limit point S' can be taken to be the limit of a subsequence Λ of $\mathbb{N}$. Since $\pi(S)$ and $\pi(T_z)$ commute in the Calkin algebra, $\pi(S)$ commutes with $\pi(T_{\bar{z}}) = \pi(T_z)^{-1}$. Hence S commutes modulo compacts with $T_{\bar{z}}$, and, therefore, also with $\mathscr{T}(C)$, the C^* algebra generated by T_z.

If K is compact, $\lim_{n \to \infty} T_{\bar{z}^n} K T_{z^n} = 0$ in norm since $T_{z^n} \to 0$ in the weak operator topology. Let b be a continuous inner function. Then,

$$\begin{aligned} S' - T_{\bar{b}} S' T_b &= w^* \lim_{n \in \Lambda} (S - T_{\bar{z}^n} S T_{z^n}) - T_{\bar{b}}(S - T_{\bar{z}^n} S T_{z^n})\, T_b \\ &= w^* \lim_{n \in \Lambda} (S - T_{\bar{b}} S T_b) - T_{\bar{z}^n}(S - T_{\bar{b}} S T_b)\, T_{z^n}. \end{aligned}$$

But, $S - T_{\bar{b}} S T_b = T_{\bar{b}}(T_b S - S T_b)$ is compact, so the second term tends to zero in norm. So, $S' - T_{\bar{b}} S' T_b = S - T_{\bar{b}} S T_b = \sigma_b$. In particular, for $b = z$, $S - S' = T_{\bar{z}}(S - S') T_z$. This is the functional equation determining the Toeplitz operators [1]. So there is a function f in L^∞ such that $S = T_f + S'$.

Now it follows that $T_f = w^* \lim_{n\in\Lambda} T_{\bar{z}^n} S T_{z^n}$. Let h_1, h_2 be functions in $H^\infty \cap C$. Then,

$$\begin{aligned} w^* \lim_{n\in\Lambda} T_{\bar{z}^n} T_{\bar{h}_1 h_2} S T_{z^n} &= w^* \lim_{n\in\Lambda} T_{\bar{z}^n} T_{\bar{h}_1}[S T_{h_2} + D(T_{h_2})]\, T_{z^n} \\ &= w^* \lim_{n\in\Lambda} [T_{\bar{h}_1} T_{\bar{z}^n} S T_{z^n} T_{h_2} + T_{\bar{z}^n} T_{\bar{h}_1} D(T_{h_2})\, T_{z^n}] \\ &= T_{\bar{h}_1} T_f T_{h_2} = T_{\bar{h}_1 h_2 f}, \qquad \text{since } D(T_{h_2}) \text{ is compact.} \end{aligned}$$

Since $\{\bar{h}_1 h_2 : h_1, h_2 \in H^\infty \cap C\}$ is dense in C, it follows that

$$w^* \lim_{n\in\Lambda} T_{\bar{z}^n} T_g S T_{z^n} = T_{gf}$$

for all continuous functions g.

If S' is compact, then $S = T_f + S'$ and $\sigma_n = S' - T_{\bar{z}^n} S' T_{z^n}$. Hence by the above remarks, we have $\lim_{n\to\infty} \sigma_n = S'$ in norm.

The proofs of (4′) and (5′) are identical to the calculations for (4) and (5). In (5′), we note that $\{\bar{h}_1 h_2 : h_1, h_2 \in H^\infty\}$ is dense in L^∞. If $S = T_f + X$ with X in the commutator ideal of $\mathscr{T}(L^\infty)$, then $\sigma_\omega = X - T_{\bar{\omega}} X T_\omega$. By Douglas [4], for any $\epsilon > 0$, there is an inner function ω such that $\| X T_\omega \| < \epsilon$. Hence $\lim_{\omega\in\Sigma} \sigma_\omega = X$.

The next lemma is a basic element in the proof of main theorem. Let $\mathscr{PC}$ be the space of piecewise continuous functions in L^∞. Consider the function F: $\mathscr{PC} \to \mathscr{L}(H^2)$ defined by $F(\chi) = T_\chi S - T_{\chi f}$, where f is defined in terms of S as in Lemma 1. It is clear that if g_n are in $\mathscr{PC}$, uniformly bounded in the sup norm, such that $g_n \to g_0$ pointwise, then $F(g_n) \to F(g_0)$ in the strong operator topology. We have $F(1) = S - T_f = S'$, which is not compact unless S is a compact perturbation of a Toeplitz operator. Keep this function in mind to motivate the following lemma.

LEMMA 2. *Suppose F: $\mathscr{PC} \to \mathscr{L}(H^2)$ is a linear map such that:*

(P1) *If g_n are in $\mathscr{PC}$ with $\| g_n \|_\infty \leqslant M$ and $g_n \to g_0$ pointwise, then $w - \lim F(g_n) = F(g_0)$ in the weak operator topology;*

(P2) *$F(1)$ is not compact, say $\| \pi F(1) \| > \alpha > 0$;*

(P3) *if f, g are in $\mathscr{PC}$ and have disjoint closed supports, then $F(f)\, F(g)$ and $F(f)\, F(g)^*$ are compact.*

Then, there exist characteristic functions $\{\chi_n : n \geqslant 1\}$ in $\mathscr{PC}$ of disjoint closed support such that $\| F(\chi_n) \| > \alpha/4$. These sets can be chosen to cluster at only one point.

Consequently, there exist trigonometric polynomials h_n such that $\| h_n \| \leqslant 2$, $\| h_n(1 - \chi_n) \| \leqslant 2^{-n}$, and $\| F(h_n) \| > \alpha/4$.

Remark. Since χ_n is in $\mathscr{PC}$ it is the finite union of closed intervals. We can suppose, in fact, that χ_n is a closed interval if we change the constant to $\alpha/8$.

Proof. Let $\mathscr{E}$ be the collection of all characteristic functions χ of *closed intervals* such that $\|F(\chi)\| > \alpha/4$. Let

$$E = \bigcap_{n\geqslant 1} \left(\bigcup \{\chi \in \mathscr{E} : |\chi| \leqslant 1/n\}\right)^{\text{cl}}.$$

Here $|\chi|$ is the linear measure of χ as a subset of the circle. We claim that E is nonempty. For if E is empty, then, since it is the intersection of nested closed sets, one of these sets is empty. That is, there exists an integer n such that $|\chi| \leqslant 1/n$ implies that $\|F(\chi)\| \leqslant \alpha/4$.

Divide $\partial\mathscr{D}$ into an even number of closed intervals χ_i, $i = 1,..., 2k$, which are disjoint except for their endpoints, so that $\partial\mathscr{D} = \bigcup \chi_i$. Each of the two collections $\{\chi_i : i \text{ is odd}\}$, $\{\chi_i : i \text{ is even}\}$ consist of mutually disjoint closed intervals. We have

$$\alpha < \|\pi F(1)\| \leqslant \tfrac{1}{2}\|\pi(F(1) + F(1)^*)\| + \tfrac{1}{2}\|\pi(F(1) - F(1)^*)\|.$$

Therefore, for $\epsilon = 1$ or -1, we have $\|\pi(F(1) + \epsilon F(1)^*)\| > \alpha$. Let $A = \pi(F(1) + \epsilon F(1)^*)$. Let $A_i = \pi(F(\chi_i) + \epsilon F(\chi_i)^*)$. By (P3), if χ_i, χ_j are disjoint, then $A_i A_j = 0$. Hence $\{A_i : i \text{ odd}\}$ (respectively, even) is a finite collection of normal, commuting, mutally annihilating operators. Therefore, by a simple estimate on the spectral radius, we get

$$\left\|\sum_{i\,\text{odd}} A_i\right\| = \max_i \|A_i\| \leqslant 2 \max \|F(\chi_i)\| \leqslant \alpha/2.$$

The analogous inequality holds for i even. Now by the linearity of F, $\sum_{i=1}^{2k} A_i = A$. So we have

$$\alpha < \|A\| = \left\|\sum A_i\right\| \leqslant \left\|\sum_{\text{odd}} A_i\right\| + \left\|\sum_{\text{even}} A_i\right\| \leqslant \alpha.$$

This contradiction shows that E is nonempty.

Let $x_0 \in E$. We proceed by induction. Suppose we have chosen disjoint characteristic functions $\chi_i \in \mathscr{E}$, $i = 1,..., n$, with $\alpha_n = d(x_0, \bigcup \chi_i) > 0$. Since $x_0 \in E$, we have

$$x_0 \in \left(\bigcup \{\chi \in \mathscr{E} : |\chi| \leqslant \alpha_n/3\}\right)^{\text{cl}}.$$

Hence, there exists a μ in $\mathscr{E}$, $|\mu| \leqslant \alpha_n/3$ such that $d(x_0, \mu) < \alpha_n/3$. So, $d(\mu, \bigcup \chi_i) \geqslant \alpha_n/3$. If x_0 does not belong to μ, let $\chi_{n+1} = \mu$. Otherwise, set $\mu_t = \mu \cap \{x : d(x, x_0) \geqslant t\}$. Then $\mu_t \to \mu$ pointwise, so, by (P1), $F(\mu_t) \to F(\mu)$

in the weak operator topology. Since the norm is lower semicontinuous in the weak operator topology, and $\|F(\mu)\| > \alpha/4$, we see that there exists a $t > 0$ such that $\|F(\mu_t)\| > \alpha/4$. Let $\chi_{n+1} = \mu_t$. Then $\alpha_{n+1} = d(x_0, \bigcup_{i=1}^{n+1} \chi_i) \geqslant t > 0$. It is clear that the sets $\{\chi_n\}$ cluster only at x_0. We remark that the sets μ_t may be the union of two intervals, say μ^+ and μ^-. Then since $F(\mu_t) = F(\mu^+) + F(\mu^-)$, we can chose one of these with norm greater than $\alpha/8$.

Fix $\chi = \chi_n$, and choose continuous function g_i such that $0 \leqslant g_i \leqslant \chi$ and $g_i \to \chi$ pointwise. We argue as above to find an integer i such that $\|F(g_i)\| > \alpha/4$. Let k_j be the jth Fejer mean of g_i. Then $\| k_j - g_i \|_\alpha$ tends to zero as j tends to infinity, so again by the above argument we choose an integer j such that $\|F(k_j)\| > \alpha/4$ and also $\| k_j - g_i \| < 2^{-n}$. For $\chi = \chi_n$, let $h_n = k_j$. We compute

$$\| h_n \| \leqslant \| g_i \| + 2^{-n} \leqslant 2,$$
$$\| h_n(1 - \chi_n)\| \leqslant \| g_i(1 - \chi_n)\| + 2^{-n} = 2^{-n}.$$

Hence we see that the functions h_n satisfy the requirements of the lemma.

LEMMA 3. *Let $\mathfrak{A}$ be a weakly closed subalgebra of $\mathscr{L}(H^2)$. Let S be an operator on H^2 and D: $\mathfrak{A} \to \mathscr{L}(H^2)$ be the derivation $D(a) = aS - Sa$. Suppose there exist $\delta > 0$, $M > 0$, and elements a_n in $\mathfrak{A}$ such that $\| D(a_n)\| > \delta > 0$ and $\| \sum_{n\in J} a_n \| \leqslant M$ for all finite subsets J of $\mathbb{N}$. Then, there exists an element b in $\mathfrak{A}$ such that $D(b)$ is not compact.*

Proof. We can assume that $D(a_n)$ is compact for all n. We claim that $a_n \to 0$ in the strong operator topology. If not, there is a unit vector h in H^2 such that $k_n = a_n h$ has $\| k_n \| \geqslant \delta$, for all n in an infinite set J. It is an elementary exercise, left to the reader, to show that we can find a finite subset J' of J so that $\| \sum_{n\in J'} k_n \| > M$. This contradicts $\| \sum_{n\in J'} a_n \| \leqslant M$.

Let $\{z_n : n \geqslant 0\}$ be an orthonormal basis for H^2. Let R_n be the orthogonal projection onto the span of $\{z_0, ..., z_n\}$. Now since $a_n \to 0$ strongly, we also have $D(a_n) \to 0$ strongly. Using this fact and the compactness of $D(a_n)$, we can inductively choose a subsequence Γ of $\mathbb{N}$ and corresponding projections Q_k which are finite dimensional and mutually orthogonal. These will be chosen so that for n_k in Γ, we have

(1) $\| Q_k D(a_{n_k}) Q_k \| > \delta$,

(2) $\| D(a_{n_k})(I - Q_k)\| < 3^{-k}\delta$,

(3) $\| R_k a_{n_k} R_k \| < 2^{-k}$.

If we have chosen $n_1, ..., n_k$ and $Q_1, ..., Q_k$, let $Q = \sum Q_1$. Since Q is finite dimensional and $D(a_n) \to 0$ strongly, we can find an a_n such that $\| R_{k+1} a_n R_{k+1} \| < 2^{-k+1}$ and $\| D(a_n) Q \| < 3^{-k-2}\delta$. Then since $D(a_n)$ is compact, we can choose Q_{k+1} orthogonal to Q, finite dimensional, so that (1) and (2) are satisfied. Set $a_{n_{k+1}} = a_n$.

For convenience, we relabel so that $a_k = a_{n_k}$. Let $b_k = \sum_{n=1}^{k} a_n$. If h_1, h_2 are in $R_n H$ and $k \geqslant l \geqslant n$, then

$$|((b_k - b_l) h_1, h_2)| \leqslant \sum_{l+1}^{k} |(a_i h_1, h_2)| \leqslant \sum_{l+1}^{k} \| R_n a_i R_n \| \| h_1 \| \| h_2 \|$$
$$\leqslant \sum_{l+1}^{k} 2^{-i} \| h_1 \| \| h_2 \| < 2^{-l} \| h_1 \| \| h_2 \|.$$

Since $\bigcup_n R_n H^2$ is dense in H^2, and $\| b_k \| \leqslant M$ for every k, we conclude that the sequence $\{b_k\}$ converges weakly to an element b in $\mathfrak{A}$. Hence $D(b_k)$ converges weakly to $D(b)$. Therefore, since the Q_n are finite dimensional, we have

$$\lim_{k\to\infty} Q_n D(b_k) Q_n = Q_n D(b) Q_n \qquad \text{in norm.}$$

Hence,

$$\| Q_n D(b) Q_n \| = \lim_{k\to\infty} \left\| Q_n \sum_{1}^{k} D(a_i) Q_n \right\|$$
$$\geqslant \lim_{k\to\infty} \| Q_n D(a_n) Q_n \| - \sum_{i\neq n} \| Q_n D(a_i) Q_n \|.$$

But if $i \neq n$,

$$\| Q_n D(a_i) Q_n \| = \| Q_n [D(a_i)(I - Q_i)] Q_n \| < 3^{-i}\delta.$$

So,

$$\| Q_n D(b) Q_n \| \geqslant \lim_{k\to\infty} \delta - \sum 3^{-i}\delta \geqslant \delta/2.$$

This is true for all n, and the projections $\{Q_n\}$ are nonzero and mutually orthogonal. It follows that $D(b)$ is not compact.

Proof of Theorem 2. We are now ready to complete the proof of our main theorem. Suppose S in $\mathscr{L}(H^2)$ is not the sum of a Toeplitz operator and a compact operator. We suppose that $T_z S - S T_z$ is compact, for otherwise we can take $h = z$. By Lemma 1, we choose a subsequence Λ of $\mathbb{N}$, and a function f in L^∞ such that

$$S' = S - T_f = w^* \lim_{n\in\Lambda} S - T_{\bar{z}^n} S T_{z^n}.$$

The operator S' is not compact, say $\| \pi(S') \| > \alpha > 0$.

We apply Lemma 2 to the map $F(g) = T_g S - T_{gf}$. Because of the remarks preceding Lemma 2, we need only show that F satisfies P3. Let f_1, f_2 be piecewise continuous with disjoint closed supports. Let g_1, g_2 be functions in C such that $g_i \equiv 1$ on the support of f_i, and $g_1 g_2 = 0$. By Lemma 1, πT_{g_i} commutes with πS, and by [4], we have for every h in L^∞,

$$\pi(T_{g_i}) \pi(T_h) = \pi(T_{g_i h}) = \pi(T_h) \pi(T_{g_i}).$$

It follows that

$$\pi(T_{g_i})\,\pi(F(f_i)) = \pi F(f_i) = \pi(F(f_i))\,\pi(T_{g_i}),$$

and the analogous relation holds for $\pi F(f_i)^*$. Thus we have

$$\begin{aligned}\pi F(f_i)\cdot\pi F(f_2) &= \pi F(f_1)\cdot\pi T_{g_1}\cdot\pi T_{g_2}\cdot\pi F(f_2)\\ &= \pi F(f_1)\cdot\pi T_{g_1g_2}\cdot\pi F(f_2) = 0.\end{aligned}$$

Hence $F(f_1)F(f_2)$ is compact, and similarly $F(f_1)F(f_2)^*$ is compact. So, from Lemma 2, there exist trigonometric polynomials h_n and characteristic functions χ_n of disjoint closed sets such that $\|h_n\| \leqslant 2$, $\|h_n(1-\chi_n)\| \leqslant 2^{-n}$, and $\|F(h_n)\| > \alpha/4$.

We compute for $h = h_n$

$$\begin{aligned}T_{\bar z^k}T_hD(T_{z^k}) &= T_{\bar z^k}T_h(T_{z^k}S - ST_{z^k})\\ &= T_hS - T_{\bar z^k}T_hST_{z^k}.\end{aligned}$$

From the derivation identity, we have

$$D(T_{z^kh}) = D(T_hT_{z^k}) = T_hD(T_{z^k}) + D(T_h)\,T_{z^k}.$$

We get

$$\begin{aligned}w^*\lim_{k\in\Lambda} T_{\bar z^k}D(T_{z^kh}) &= w^*\lim_{n\in\Lambda}(T_hS - T_{\bar z^k}T_hST_{z^k}) + T_{\bar z^k}D(T_h)\,T_{z^k}\\ &= T_hS - T_{hf} \qquad \text{by Lemma 1.}\end{aligned}$$

By lower semicontinuity of the norm, and the inequality $\|T_{h_n}S - T_{h_nf}\| = \|F(h_n)\| > \alpha/4$, we can choose an integer k in Λ such that $\|T_{\bar z^k}D(T_{z^kh_n})\| > \alpha/4$ and $p_n = z^kh_n$ belongs to H^∞. We then have $\|D(T_{p_n})\| \geqslant \|T_{\bar z^k}D(T_{p_n})\| > \alpha/4$, and

$$\|p_n\| = \|z^kh_n\| \leqslant 2, \qquad \text{and} \qquad \|p_n(1-\chi_n)\| = \|z^kh_n(1-\chi_n)\| \leqslant 2^{-n}.$$

Now, if J is a finite subset of $\mathbb{N}$, let $p_J = \sum_{n\in J} p_n$. Then

$$\begin{aligned}\|p_J\chi_m\|_\infty &\leqslant \sum_{n\in J}\|p_n\chi_m\| \leqslant \|p_m\| + \sum_{n\neq m}\|p_n(1-\chi_n)\|\\ &\leqslant 2 + \sum_n 2^{-n} = 3.\end{aligned}$$

If $\mu = 1 - \sum_m \chi_m$, then $\|p_J\mu\| \leqslant \sum\|p_n\mu\| \leqslant \sum 2^{-n} = 1$. Hence $\|p_J\| \leqslant 3$ for all finite subsets J.

Therefore we can apply Lemma 3, with $a_n = T_{p_n}$ and $\mathfrak{A} = \mathscr{T}(H^\infty)$. This gives us a function h in H^∞ such that $D(T_h)$ is not compact.

We have $T_h = w - \lim T_{b_k}$ where $b_k = \sum_{i=1}^{k} p_{n_i}$. The set $\{x: |p_n(x)| \geq 2^{-n}\}$ is contained in χ_n, and the sets $\{\chi_n\}$ cluster only at the point x_0. So the sequence $\{b_k\}$ converges uniformly on sets bounded away from x_0 to a function which is continuous except at x_0. Hence h is continuous except at x_0. This concludes the proof.

Proof of Theorem 1. We will now prove the results stated in the first section. If S commutes modulo the compacts with all analytic Toeplitz operators, it follows from Theorem 2 that S has the form $S = T_f + K$, where K is compact. Therefore, we see from Sarason [8] that f is in $H^\infty + C$.

We remark that if S is in the Toeplitz algebra, this result follows in a more elementary way. For then $S = T_f + X$, where X is in the commutator ideal of $\mathscr{T}(L^\infty)$. We have that $\sigma_\omega = T_{\bar{\omega}}(T_\omega S - ST_\omega)$ is compact for every inner function ω. So be Lemma 1, $X = \lim_{\omega \in \Sigma} \sigma_\omega$ in norm, and hence X is compact. We now apply Sarason's result as above.

To prove Corollary 1, we note that $\mathscr{T}(L^\infty)$ is generated by $\mathscr{T}(H^\infty)$ and $\mathscr{T}(\overline{H^\infty})$. Hence

$$\mathscr{T}(L^\infty)^{\mathrm{ec}} = \mathscr{T}(H^\infty)^{\mathrm{ec}} \cap \mathscr{T}(\overline{H^\infty})^{\mathrm{ec}} = \mathscr{T}(H^\infty + C) \cap \mathscr{T}(\overline{H^\infty + C}) = \mathscr{T}(QC).$$

Corollary 2 is immediate from Theorem 1 and the fact that $\pi\mathscr{T}(H^\infty + C)$ is abelian.

COROLLARY 3. *If an operator S is not in $\mathscr{T}(H^\infty + C)$, then there is an inner function ω such that $ST_\omega - T_\omega S$ is not compact.*

Proof. By Theorem 1, there is an analytic function h such that $ST_h - T_h S$ is not compact. A theorem of Marshall [10] shows that the linear span of the inner functions is norm dense in H^∞. The set of noncompact operators is open, so we can approximate h by a finite linear combination of inner functions $\sum \alpha_i \omega_i$ so that $\sum \alpha_i (ST_{\omega_i} - T_{\omega_i} S)$ is not compact.

Remark. Marshall's results actually say more. If we take h to be continuous except at x_0, we can approximate h in norm by Blaschke products which are continuous except at x_0. So if S is not a Toeplitz operator plus a compact, we can find a Blaschke product b with zeros accumulating only at x_0 for which $ST_b - T_b S$ is not compact.

DEFINITION. A derivation D of an algebra $\mathfrak{A}$ into itself is inner if $D(X) = XS - SX$ for some S in $\mathfrak{A}$.

COROLLARY 4. *Every derivation D of $\mathscr{T}(H^\infty + C)$ into $\mathscr{L}\mathscr{C}(H^2)$ is inner.*

Proof. The operator D restricted to $\mathscr{L}\mathscr{C}(H^2)$ is a derivation of the compacts into themselves. It is well known [2] that every derivation on the compacts has the form $D(X) = XS - SX$ for some S in $\mathscr{L}(H^2)$. If A is in $\mathscr{T}(H^\infty + C)$ and K is compact, then

$$\begin{aligned} D(AK) = AKS - SAK &= (AKS - ASK) + (AS - SA)K \\ &= AD(K) + D(A)K = AKS - ASK + D(A)K. \end{aligned}$$

Therefore $D(A)K = (AS - SA)K$ for every compact operator K. Hence $D(A) = AS - SA$. Since S commutes with all A in $\mathscr{T}(H^\infty + C)$ modulo the compacts, we have that S is in $\mathscr{T}(H^\infty + C)$ by Theorem 1.

An immediate consequence of this is the following.

COROLLARY 5. *Every derivation of $\mathscr{T}(L^\infty)$ into the compact operators is of the form $D(X) = XS - SX$ with S in $\mathscr{T}(QC)$.*

We consider the matrix-valued case. The operator algebra $\mathscr{T}(L^\infty) \otimes M_n$ acts on $H^2 \otimes \mathbb{C}^n$, M_n is the $n \times n$ matrix algebra over $\mathbb{C}$. A general reference for this is Douglas [5].

COROLLARY 6. *An operator S in $\mathscr{L}(H^2 \otimes \mathbb{C}^n)$ commutes modulo the compacts with all operators in $\mathscr{T}(H^\infty) \otimes M_n$ if and only if $S = T_f \otimes I_n + K$, where f is in $H^\infty + C$ and K is compact.*

Proof. Let δ_{ij} be the $n \times n$ matrix zero everywhere except for a 1 in the (i, j) entry. A simple computation of $D(T_h \otimes \delta_{ij})$ for h in H^∞ shows that S has the desired form.

COROLLARY 7. *An operator S in $(H^2 \otimes \mathbb{C}^n)$ is in the essential commutant of $\mathscr{T}(L^\infty) \otimes M_n$ if and only if $S = T_f \otimes I_n + K$, where f is in QC and K is compact.*

THEOREM 3. *Let α be an automorphism of $\mathscr{T}(H^\infty) + \mathscr{LC}(H^2)$. Then α is spatial, and has the factorization $\alpha = \alpha_1\alpha_2$, where*

(1) *$\alpha_1(T_h) = T_{h\circ b}$ for a Blaschke factor $b = \lambda(z - a)/(1 - \bar{a}z)$, $|a| < 1$, and $|\lambda| = 1$. More generally, $\alpha_1(A) = U_1{}^*AU_1$ for all A in $\mathscr{T}(H^\infty) + \mathscr{LC}(H^2)$, where $U_1{}^*f = e(f \circ b)$ for f in H^2 and e a unit vector in $H^2 \ominus bH^2$.*

(2) *$\alpha_2(A) = U_2{}^*AU_2$ for a unitary U_2 in $\mathscr{T}(QC)$. So $U = T_g + K$, where g is unimodular in QC and K is compact.*

Proof. Since $\mathscr{LC}(H^2)$ is the unique minimal closed two-sided ideal in $\mathscr{T}(H^\infty) + \mathscr{LC}(H^2)$, we must have $\alpha(\mathscr{LC}(H^2)) = \mathscr{LC}(H^2)$. So, by a well-known theorem [3], there is a unitary operator W such that $\alpha(K) = W^*KW$ for all K in $\mathscr{LC}(H^2)$. If A is in $\mathscr{T}(H^\infty) + \mathscr{LC}(H^2)$, then $(W^*AW)(W^*KW)$ $W^*AKW = \alpha(AK) = \alpha(A)\,\alpha(K) = \alpha(A)W^*KW$ for all compact operators K. Hence $\alpha(A) = W^*AW$.

There is a natural map from the automorphisms of $\mathscr{T}(H^\infty) + \mathscr{LC}(H^2)$ onto the automorphisms of H^∞ by projecting into the Calkin algebra. The algebras H^∞, $\mathscr{T}(H^\infty)$, and $\pi\mathscr{T}(H^\infty)$ are isometrically isomorphic as Banach algebras, so we can identify them here. The automorphisms of H^∞ are known [9, p. 143] to be of the form $\alpha(h) = h \circ b$, where b is a conformal map of the disc onto itself.

The kernel of this map is the set of automorphisms α such that $\alpha(T_h) = T_h + K$, with K compact. Since α is spatial, it is induced by a unitary operator U. So U and U^* essentially commute with $\mathscr{T}(H^\infty)$. Hence U belongs to $\mathscr{T}(QC)$.

We now show that automorphisms of $\mathscr{T}(H^\infty)$ are spatial. Let b be a conformal automorphism of the disc. Let e be a unit vector in $H^2 \ominus bH^2$. Define an operator on H^2 by $U_1^*f = e(f \circ b)$ for f in H^2. Since e is in H^∞, it follows that $e(f \circ b)$ is in H^2. A computation shows that $| db/dz | = | e |^2$. So $\| U_1^*f \|^2 = \int | e(f \circ b) |^2 dz = \int | f |^2 \circ b \, | db/dz | \, dz = \int | f |^2 dz = \| f \|_2^2$. The operator U_1^* is clearly invertible, hence it is unitary on H^2. If h is in H^∞,

$$U_1^* T_h U_1 e(f \circ b) = U_1^* T_h f = e(fh \circ b) = (h \circ b)\, e(f \circ b) = T_{h \circ b}\, e(f \circ b).$$

Hence, $U_1^* T_h U_1 = T_{h \circ b}$.

Let α be an automorphism of $\mathscr{T}(H^\infty) + \mathscr{LC}(H^2)$. Then $\pi\alpha$ is an automorphism of H^∞. This lifts to a spatial automorphism of $\mathscr{T}(H^\infty)$, $\alpha_1(A) = U_1^* A U_1$. Let $\alpha_2 = \alpha_1^{-1}\alpha$. Since $\pi\alpha_2$ is the identity, we have $\alpha_2(A) = U_2^* A U_2$, for some unitary U_2 in $\mathscr{T}(QC)$.

Note added in proof. Theorem 3 is also valid for the automorphisms of $\mathscr{T}(H^\infty + C)$.

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References

1. A. Brown and P. R. Halmos, Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.* **213** (1963/64), 89–102.
2. J. Dixmier, "Les Algèbres d'Opérateurs dans l'Espace Hilbertien," Gauthier–Villars, Paris, 1957.
3. J. Dixmier, "Les C*-Algèbres et Leurs Représentations," Gauthier–Villars, Paris, 1964.
4. R. G. Douglas, "Banach Algebra Techniques in Operator Theory," Academic Press, New York, 1972.
5. R. G. Douglas, "Banach Algebra Techniques in the Theory of Toeplitz Operators," CMBS Reg. Conf., No. 15, Amer. Math. Soc., Providence, R. I., 1973.

6. R. G. Douglas, Local Toeplitz operators, to appear.
7. B. E. Johnson and S. K. Parrot, Operators commuting with a von Neumann algebra modulo the compact operators, *J. Functional Analysis* **11** (1972), 39–61.
8. D. E. Sarason, On products of Toeplitz operators, *Acta Math. (Szeged)* **34** (1972).
9. K. Hoffman, "Banach Spaces of Analytic Functions," Prentice–Hall, Englewood Cliffs, N.J., 1962.
10. D. E. Marshall, Blaschke products generate H^∞, *Bull. Amer. Math. Soc.* **82** (1976), 494–496.

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